Odd Perfect Numbers?

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M.A. Conference Session

2016

A *perfect* number is a positive integer whose factor sum (excluding itself) is equal to itself*.

e.g. The factors of 6 are 1, 2 and 3, and 1+2+3=6

An *abundant* number is a positive integer whose factor sum is greater than itself.

e.g. The factors of 12 are 1, 2, 3, 4 and 6,
and 1+2+3+4+6=16 > 12

A *deficient* number is a positive integer who factor sum is less than itself.

e.g. The factors of 15 are 1, 3 and 5
and 1+3+5=9 < 15

*Until page 5, to avoid repetition, in using the terms *factor* and *factor sum* I will exclude the number itself.

The three smallest perfect numbers are:

- 6 = 1 + 2 + 3
- 28 = 1 + 2 + 4 + 7 + 14
- 496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248

The next two are 8128 and 33550336.

The largest known perfect number (discovered this year)

is $2^{74207280}$ $(2^{74207281} - 1)$.

This is approximately equal to 4.5 x $10^{44677234}$

However, the 49 known perfect numbers are all even and so a big unsolved problem is

'Do odd perfect numbers exist?'

What are the factor sums of

a) 36
b) 120 ?

At Key Stage 3 a method for solving this would be to test for divisibility starting with 1 and writing down factor pairs so discovered until you reach the square root of the number.

Hence

a) 1, (36), 2, 18, 3, 12, 4, 9, 6.

Hence the factor sum of 36 is 55.

b) 1, (120), 2, 60, 3, 40, 4, 30, 5, 24, 6, 20, 8, 15, 10, 12

Hence the factor sum of 120 is 240.

On the last page we had the factor sum of 120 as being 240.

120 is the smallest *triperfect* number.

A *triperfect* number is defined to be a positive integer whose factor sum is equal to three times the original number.

(The next triperfect numbers are 672 and 523776. After this there are none under 100 million – I have checked this but not by hand!)

However, you notice that the *factor sum* here now includes the number itself!

We will redefine *perfect* to be consistent with this:
N is perfect if the sum of all its divisors is 2N

$$\text{i.e. } \sigma\,(N) = 2N$$

1	2	4	7 (x1)
3	6	12	(x3)
9	18	36	(x9)
			(x13)

$$\sigma(36) = 7 \times 13 = 91$$

This is OK if there are two distinct prime factors, but what do we do if there are more?

Let us consider 120 again.

$120 = 2^3 \times 3 \times 5$

Effectively we have two layers of a cuboid.

1	2	4	8	15 (x1)
3	6	12	24	(x3)
				(x4)
5	10	20	40	(x5)
15	30	60	120	(x15)
				(x20) **(x24)**

$$\sigma(120) = 15 \times 4 \times 6 = 360$$

Note: This is the same as (1 + 2 + 4 + 8) (1 + 3) (1 + 5)

So in general if N = $p_1^{i_1} p_2^{i_2} \dots p_k^{i_k}$

$$\boldsymbol{\sigma}(\mathbf{N}) = (\mathbf{1} + \mathbf{p_1} + \mathbf{p_1}^2 + \dots + \mathbf{p_1}^{i_1})(\mathbf{1} + \mathbf{p_2} + \mathbf{p_2}^2 + \dots + \mathbf{p_2}^{i_2}) \dots (\mathbf{1} + \mathbf{p_k} + \mathbf{p_k}^2 + \dots + \mathbf{p_k}^{i_k})$$

Example

$\sigma\,(21000)$

= $\sigma\,(21 \times 1000)$

$= \sigma\,(2^3 \times 3 \times 5^3 \times 7)$

= (1 + 2 + 4 + 8) (1 + 3) (1 + 5 + 25 + 125) (1 + 7)

= $15 \times 4 \times 156 \times 8$

= 74880

So 21000 is clearly abundant!

Now in the Core 2 module sixth formers learn that

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Setting a = 1 and r = p we have that

$$1 + p + p^2 + \ldots + p^k = \frac{p^{k+1} - 1}{p - 1}$$

Consequently

$$\sigma(N) = \left(\frac{p_1^{i_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{i_2+1} - 1}{p_2 - 1}\right) \ldots \left(\frac{p_k^{i_k+1} - 1}{p_k - 1}\right)$$

Now for odd perfect numbers the sum of all its factors is equal to twice the original number. Algebraically this can be expressed as

$$2p_1^{i_1} p_2^{i_2} \ldots p_k^{i_k} = \left(\frac{p_1^{i_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{i_2+1} - 1}{p_2 - 1}\right) \ldots \left(\frac{p_k^{i_k+1} - 1}{p_k - 1}\right)$$

Note the LHS is divisible by 2 but NOT divisible by 4

This means that the RHS also has to be 2 modulo 4.

We therefore need to consider products within modulo 4.

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$$2p_1^{i_1}\, p_2^{i_2} \ldots p_k^{i_k} = \left(\frac{p_1^{i_1+1} - 1}{p_1 - 1}\right)\left(\frac{p_2^{i_2+1} - 1}{p_2 - 1}\right) \ldots. \left(\frac{p_k^{i_k+1} - 1}{p_k - 1}\right)$$

If all of the expressions in brackets equal 1 or 3 (mod 4) the product will be 1 or 3 (mod 4) too and so the RHS will not equal the 2 (mod 4) required.

However, if any of the expressions equal 0 (mod 4) or more than one of them equals 2 (mod 4), the product will be 0 (mod 4) and so again the RHS will not equal the 2 (mod 4) required.

We therefore need one of the prime factors to give rise to a factor sum that is 2 (mod 4) and the rest of the prime factors to give rise to factor sums of 1 or 3 (mod 4).

Let us now consider what value modulo 4 $\frac{p^{i+1} - 1}{p-1}$ has in two different cases:

Given that we are dealing with ODD perfect numbers, prime factors will either be 1 mod 4 or 3 mod 4.

<u>p ≡ 1 mod 4</u>

Highest power	1	p	p^2	p^3	p^4	p^5	p^6	p^7
Sum of $\frac{p^{i+1}-1}{p-1}$ (mod 4)	1	2	3	0	1	2	3	0

<u>p ≡ 3 mod 4</u>

Highest power	1	p	p^2	p^3	p^4	p^5	p^6	p^7
Sum of $\frac{p^{i+1}-1}{p-1}$ (mod 4)	1	0	1	0	1	0	1	0

This means that an odd perfect number can't have a prime factor raised to an odd power, *unless it is a prime 1 mod 4, raised to a power which is also 1 mod 4*.

Moreover, this condition must happen once and only once!

In summary all the indices of distinct primes are even apart from one and only one prime of the form 1 (mod 4) which has an index which is also 1 (mod 4).

Without this exception the product of all the bracketed expressions will be 1 or 3 (mod 4).

With more than one exception, two bracketed expressions of the form 2 (mod 4) will create an overall product of 0 (mod 4).

We can therefore write

$$2N = 2a^2b^{4n+1} = S\left(\frac{b^{4n+2} - 1}{b-1}\right)$$

N – odd perfect number

b – the unique prime factor raised to the power of 4n +1

a^2 – all the other prime factors combined (given they all have even indices)

S – the sum of all the factors of a^2

Could we have odd perfect numbers with only two distinct prime factors?

If so,

$2N = 2a^{2r}b^{4n+1} = (1 + a + a^2 + \dots + a^{2r})\,(1 + b + b^2 + \dots + b^{4n+1})$

where b is the unique prime factor previously mentioned and a is in this case the other prime factor.

$$2 = \frac{\sigma(a^{2r})}{a^{2r}} \times \frac{\sigma(b^{4n+1})}{b^{4n+1}}$$

$$\left(\frac{1}{a^{2r}} + \frac{1}{a^{2r-1}} + \frac{1}{a^{2r-2}} \dots + 1\right)\left(\frac{1}{b^{4n+1}} + \frac{1}{b^{4n}} + \frac{1}{b^{4n-1}} + \dots + 1\right)$$

However, $1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^i} < 1 + \frac{1}{p} + \frac{1}{p^2} + \dots = \frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$

We want

$$2 = \left(\frac{1}{a^{2r}} + \frac{1}{a^{2r-1}} + \frac{1}{a^{2r-2}} \ldots + 1\right)\left(\frac{1}{b^{4n+1}} + \frac{1}{b^{4n}} + \frac{1}{b^{4n-1}} + \ldots + 1\right)$$

However, $1 + \frac{1}{p} + \frac{1}{p^2} + \ldots + \frac{1}{p^i} < \frac{p}{p-1}$

If p = 3, $\frac{\sigma(p^i)}{p^i} < \frac{3}{2}$ where p could be a or b.

If p = 5, $\frac{\sigma(p^i)}{p^i} < \frac{5}{4}$

Other values of x will give smaller values of $\frac{\sigma(p^i)}{p^i}$.

Therefore,

$$\left(\frac{1}{a^{2r}} + \frac{1}{a^{2r-1}} + \frac{1}{a^{2r-2}} \ldots + 1\right)\left(\frac{1}{b^{4n+1}} + \frac{1}{b^{4n}} + \frac{1}{b^{4n-1}} + \ldots + 1\right) < \frac{3}{2} \times \frac{5}{4} = \frac{15}{8}$$

and so an odd number with only two distinct prime factors would be deficient and thus could not be perfect.

Could we have odd perfect numbers with only three distinct factors?

If so,

$$2{p_1}^{2r_1}{p_2}^{2r_2}b^{4n+1} = (1 + p_1 + {p_1}^2 + \ldots + {p_1}^{2r_1})(1 + p_2 + {p_2}^2 + \ldots + {p_2}^{2r_2})(1 + b + b^2 + \ldots + b^{4n+1})$$

Doing the same calculation as before, but now with three distinct prime factors, we find that $\frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} = \frac{105}{48} > 2$.

So, we can't rule this situation yet!

Which sets of prime factors could give a value of $\frac{\sigma(N)}{N}$ which reaches 2?

Below we will for the time being ignore the modulo 4 requirements and just consider whether a particular set of primes could have a chance of being anything other than deficient.

Sets of primes	Upper bound	
$\{3, 5, 7\}$	$\frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} = \frac{105}{48} > 2$	possible
$\{3, 5, 11\}$	$\frac{3}{2} \times \frac{5}{4} \times \frac{11}{10} = \frac{165}{80} > 2$	possible
$\{3, 5, 13\}$	$\frac{3}{2} \times \frac{5}{4} \times \frac{13}{12} = \frac{195}{96} > 2$	possible
$\{3, 5, 17\}$	$\frac{3}{2} \times \frac{5}{4} \times \frac{17}{16} = \frac{255}{128} < 2$	always deficient (also if 17+)
$\{3, 7, 11\}$	$\frac{3}{2} \times \frac{7}{6} \times \frac{11}{10} = \frac{231}{120} < 2$	always deficient (nor if 7+ or 3+)

We thus have four different cases to consider:

Case 1: $N = 3^{2l} \times 7^{2m} \times 5^{4n+1}$

Case 2: $N = 3^{2l} \times 11^{2m} \times 5^{4n+1}$

Case 3: $N = 3^{2l} \times 5^{2m} \times 13^{4n+1}$

Case 4: $N = 3^{2l} \times 13^{2m} \times 5^{4n+1}$

Note that in all these cases, N is divisible by 5.

This means that if an odd perfect number exists in any of these cases,
5 must divide $1 + p + p^2 + \ldots + p^i$ for some prime p and index i.

To be concrete, Case 2 would mean that:

$$2 \times 3^{2r_1} \times 11^{2r_2} \times 5^{4n+1} \text{ would be equal to}$$

$$(1 + 3 + 3^2 + \ldots + 3^{2r_1})(1 + 11 + 11^2 + \ldots + 11^{2r_2})(1 + 5 + 5^2 + \ldots + 5^{4n+1})$$

In particular, p must be 3, 7, 11 or 13, as clearly $1 + 5 + 5^2 + \ldots + 5^i$
cannot be divisible by 5 for any value of i.

We will consider on the next page a table of sums of powers in modulo 5 will help us considerably.

		p			
	p^i	1	2	3	4
	0	1	1	1	1
	1	1	2	3	4
	2	1	4	4	1
	3	1	3	2	4
i	4	1	1	1	1
	5	1	2	3	4
	6	1	4	4	1
	7	1	3	2	4
	8	1	1	1	1
	9	1	2	3	4

		p			
	σ(p^i)	1	2	3	4
	0	1	1	1	1
	1	2	3	4	0
	2	3	2	3	1
	3	4	0	0	0
i	4	0	1	1	1
	5	1	3	4	0
	6	2	2	3	1
	7	3	0	0	0
	8	4	1	1	1
	9	0	3	4	0

If p = 3, $1 + 3 + 3^2 + \ldots + 3^i$ is divisible by 5 if its index is of the form 4k+3
However, 3 needs to have an even index. Therefore 3 can't be the part divisible by 5.

If p=7, $1 + 7 + 7^2 + \ldots + 7^i$ is divisible by 5 if its index is also of the form 4k+3. (Consider the column for 2 mod 5 this time.) However, 7 also needs to have an even index. Therefore 7 can't be the part divisible by 5 either.

			p						p		
	p^i	1	2	3	4		σ(p^i)	1	2	3	4
	0	1	1	1	1		0	1	1	1	1
	1	1	2	3	4		1	2	3	4	0
	2	1	4	4	1		2	3	2	3	1
	3	1	3	2	4		3	4	0	0	0
i	4	1	1	1	1	i	4	0	1	1	1
	5	1	2	3	4		5	1	3	4	0
	6	1	4	4	1		6	2	2	3	1
	7	1	3	2	4		7	3	0	0	0
	8	1	1	1	1		8	4	1	1	1
	9	1	2	3	4		9	0	3	4	0

If p=13, given that 13 is 3 mod 5, $1 + 13 + 13^2 + \ldots + 13^i$ is divisible by 5 when the index is of the form 4k+3. However, given that an odd perfect number is of the form a^2b^{4n+1}, 13 is either part of the a and would need an even index, or is b, and then would need an index which is 1 (mod 4) rather than 3 (mod 4).

Consequently, the necessary factor of 5 can't divide into the 13 part.

Case 1: N = $3^{2l} \times 7^{2m} \times 5^{4n+1}$
Case 2: N = $3^{2l} \times 11^{2m} \times 5^{4n+1}$
Case 3: N = $3^{2l} \times 5^{2m} \times 13^{4n+1}$
Case 4: N = $3^{2l} \times 13^{2m} \times 5^{4n+1}$

If sums of powers of
$3^{2l}, 7^{2m}, 5^{4n+1}, 5^{2m}, 13^{4n+1}$ or 13^{2m}
are not divisible by 5, this rules out cases 1, 3 and 4.

This means that we are down to Case 2 if it is possible to have an odd perfect number with three distinct prime factors.

Case 2: N = $3^{2l} \times 11^{2m} \times 5^{4n+1}$

If the index of the 5 were merely 1,

$$\frac{\sigma(N)}{N} = \frac{\sigma(3^{2l})}{3^{2l}} \times \frac{\sigma(11^{2m})}{11^{2m}} \times \frac{6}{5} < \frac{3}{2} \times \frac{11}{10} \times \frac{6}{5} = \frac{198}{100} < 2$$

N will therefore be deficient and not perfect. The index of 5 in N must therefore be higher and the next possible value of 4n+1 is 5.

Therefore 1 + 11 + 11^2 + … + 11^{2m} must be divisible by 25 (although clearly it would also need to be divisible by at least 5^5). When does this occur?

index of 11, i	0	1	2	3	4	5	6	7	8	9
p^i (mod 25)	1	11	21	6	16	1	11	21	6	16
σ(p^i) (mod 25)	1	12	8	14	5	6	17	13	19	10
index of 11, i	10	11	12	13	14	15	16	17	18	19
p^i (mod 25)	1	11	21	6	16	1	11	21	6	16
σ(p^i) (mod 25)	11	22	18	24	15	16	2	23	4	20
Index of 11, i	20	21	22	23	24	25	26	27	28	29
p^i (mod 25)	1	11	21	6	16	1	11	21	6	16
σ(p^i) (mod 25)	21	7	3	9	0	1	12	8	14	5

N = $3^{2l} \times 11^{2m} \times 5^{4n+1}$ (n≥ 1)

We know that $\sigma(3^{2l})$ and $\sigma(5^{4n+1})$ can’t be divisible by 5.
Therefore if N is perfect, it is necessary for $\sigma(11^{2m})$ to be divisible by 25.

From the table we can see that this will happen when i is of the form 25j+24.
Combining this with the fact that the index of 11 is even,
the index of 11 is of the form 50j+24 !

Now $1 + 11 + 11^2 + \ldots + 11^{50j+24} = \frac{11^{50j+25} - 1}{10}$ (*)

But x – 1 divides into $x^n - 1$ for all positive integers n, and so the numerator of (*) is divisible by $11^5 - 1$.

Now $11^5 - 1 = 161050 = 2 \times 5^2 \times 3221$

But since the prime number 3221 divides $11^5 - 1$, this means that $1 + 11 + 11^2 + \ldots + 11^{50j+24}$ is divisible by the prime 3221.

Remember: in Case 2, $2 \times 3^{2r_1} \times 11^{2r_2} \times 5^{4n+1}$ would be equal to $(1 + 3 + 3^2 + \ldots + 3^{2r_1})(1 + 11 + 11^2 + \ldots + 11^{2r_2})(1 + 5 + 5^2 + \ldots + 5^{4n+1})$

However, N only has prime factors 3, 5 and 11.
So Case 2 was the last case left and this is not possible either.

Therefore an odd perfect number must have at least four distinct prime factors.